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The Method of Eigenvalleys*

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A computer-aided contour search technique for finding the complex roots of infinite- or finite-order complex transcendental equations, called the *Method of Eigenvalleys*, is presented. The method is used to solve a dispersion relation, which arises in the theory of linear wave propagation in viscous compressible liquids.

I. INTRODUCTION

Many problems in wave propagation and allied fields lead to a dispersion relation which is transcendental in the relevant complex propagation constant, and the difficulty of solving this complex eigenvalue equation has caused their solution to remain uncompleted. Typically, such problems reduce to a system of homogeneous linear equations (derived from a set of homogeneous boundary conditions), which are of the form

$$A(k)C = 0, \tag{1}$$

where $A(k)$ is a square complex coefficient matrix of order n, with usually *all* coefficients A_{ij} being functions of the complex wavenumber (alternatively called the propagation constant, separation constant, or eigenvalue) $k = k_r + jk_i$ $[j = \sqrt{(-1)}]$, and C is a complex constant matrix of order n with at least one of the elements C_i arbitrary. For a nontrivial solution of (1), the rank of $A(k)$ must be less than n, and k is found as all solutions of

$$\det(A(k)) = 0. \tag{2}$$

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Because of the complex transcendental nature of (2), obtaining the solution is generally not easy.

In the past, researchers have tried various techniques, such as expanding the transcendental functions in the coefficients of $A(k)$ in truncated series, or using directly one of many numerical minimization schemes (e.g., steepest descent, or Newton–Raphson iteration). Both procedures have their disadvantages, with the former yielding incomplete approximate eigenvalues whose associated parameter range is overly restricted by the assumption that the arguments of the truncated series must be very small (or very large for truncated asymptotic series), and with the latter being limited by both the possibilities of missing important eigenvalues and the waste of the man hours and computer time required to determine analytically, program, and compute gradients. Another technique suggested by Delves and Lyness [9] and modified in a recent work by Garg and Rouleau [10] involves a closed contour integration of (2) using Cauchy's residue theorem to isolate the number of zeroes of (2) by counting the multiples of 2π by which the phase angle of the determinant changes on the closed contour on the complex k-plane. Although a powerful technique for finding a few well separated zeroes, the method becomes very inefficient and expensive in computer time when larger numbers of more closely-spaced zeroes are encountered. The *Method of Eigenvalleys* (characteristic valleys) overcomes all of these difficulties.

II. Derivation of the Method of Eigenvalleys

Define the complex auxiliary function $E(k)$ to be

$$E(k) = \det(A(k)), \tag{3}$$

where k is an *arbitrary* complex number not necessarily an eigenvalue. Since k is complex, E will also be complex; and for k to be a solution of (2), both real and imaginary parts of E must vanish simultaneously. A much simpler equivalent condition is to require the magnitude (complex modulus) of $E(k)$ to vanish so that

$$E(k) = 0 \Leftrightarrow |E(k)| = 0. \tag{4}$$

This simplification reduces the number of variables from two dependent and two independent variables to one dependent and two independent variables. Further, $|E(k)|$ is a positive semidefinite real number, and, when k is an eigenvalue, $|E(k)|$ reduces to the dispersion relation (2). Again, the number of coordinates is reduced from four to three; and while it is impossible to *visualize* directly four coordinates, it is quite easy to picture a function represented by only three.

Hence, the *Method of Eigenvalleys* consists of first plotting level curves of the continuous characteristic surface $|E(k_r, k_i)|$, called an *eigensurface*, over some predefined region of complex k space.[1] Use of the maximum modulus principle [1] shows that, in this eigensurface, *all* concave upward regions will contain eigenvalues, since the only way a minimum of $|E(k)|$ can be obtained inside the region boundary is if $E(k)$ takes on the value zero. Since all concave regions contain eigenvalues at their base, they are called *eigenvalleys*.[2] An eigenvalley contains one and only one eigenvalue (unless the eigenvalue is repeated identically—called a degenerate solution).

Figure 1 shows a typical eigensurface, with four eigenvalleys, and the four

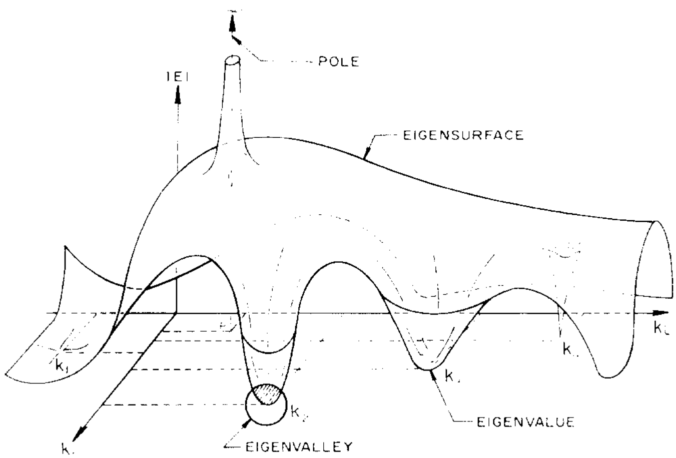

FIG. 1. Typical eigensurface of $|E| = |E(k_r, k_i)|$.

corresponding eigenvalues k_1, k_2, k_3, and k_4. Three of the eigenvalleys are well behaved, but the slope of the fourth is very steep, with a discontinuous directional derivative at its base. Such discontinuities in slope are typical and are caused by sign changes in the function $E(k)$. One pole of $E(k)$ and *no* finite hills of $|E(k)|$ are shown in Fig. 1. This is again due to the maximum modules principle [1] which shows that the only way a complex function can have a maximum on the

[1] A important restriction of the complex k space occurs in the degenerate case when k can be shown to be pure real or pure imaginary. In this instance, it is unnecessary to plot level curves of the function. Rather, a plot of $|E(k)|$ versus $k = k_r + j(0)$ and a plot of $|E(k)|$ versus $k = (0) + jk_i$ is all that is required.

[2] This definition of an eigensurface does not imply that all points on the surface are eigenvalues: eigenvalues exist only at the base of an eigenvalley. Alternatively, an eigensurface can be thought of as an error surface, since all its points represent the absolute error (positive deviation from zero) for any arbitrary choice of k.

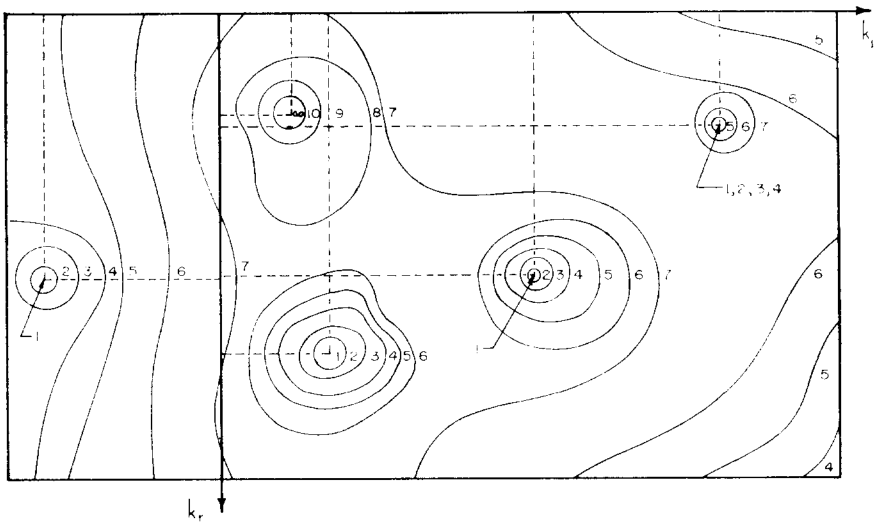

FIG. 2. Typical eigensurface contours.

interior of a closed region is if it is not analytic there; i.e., $E(k)$ is singular and possesses at least a simple pole in the interior.

Figure 2 shows a contour map of the eigensurface shown in Fig. 1. Such a map can be constructed by a digital computer and coupled to an automatic xy plotter [2, 3]. In order to distinguish the eigenvalleys from poles, the contour lines are numbered consecutively starting from the lowest elevation.

Once the coordinates of all eigenvalleys in the plotting region are visually located, the exact location of the base of the eigenvalley can be computed again using a digital computer. The actual minimization algorithm that can be used is limited by the possible discontinuous directional derivative at the base of the eigenvalley and the nonlinear complex transcedental nature of the eigenvalue equation. Pattern search [3, 4] is recommended for these reasons, since it is designed to work on highly nonlinear objective functions, and it stays on the bottom of steep valleys while searching for a minimum.[3]

III. Sample Solution

The Method of Eigenvalleys is now applied to the solution of the dispersion relation associated with steady-periodic wave propagation in a compressible viscous fluid in a rigid impermeable cylindrical tube [3, Chapter III; 7].

[3] A useful extension of the Method of Eigenvalleys is to run representative eigensurfaces and then use these as starting solutions for a gradient technique, such as the one discussed by Tiersten [11] to get neighboring solutions.

The relevant set of complex homogeneous linear equations resulting from the appropriate solution of the linear acoustic equations is given by

$$\begin{bmatrix} MJ_1(M) & kJ_1(A) \\ kJ_0(M) & AJ_0(A) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5}$$

where

$$M = \left[k^2 + \frac{F^2}{1 + j(4/3)\,FD}\right]^{1/2} \tag{6}$$

and

$$A = [k^2 - j(F/D)]^{1/2} \tag{7}$$

(where C_1 and C_2 are complex constants, $k = k_r + jk_i$ is the dimensionless propagation constant, J_0 and J_1 are the ordinary zeroth and first-order complex Bessel functions of the first kind, F is the given real dimensionless frequency, and D is the given real dimensionless first coefficient of viscosity). To obtain (5), solutions of the form

$$f(\mathscr{R}, Z, T) = \mathrm{Re}(f(\mathscr{R})\, e^{jFT+kZ}) \tag{8}$$

are assumed [where $\mathscr{R}$ is the dimensionless radius ($0 \leqslant \mathscr{R} \leqslant 1$), T the dimensionless time, and Z the dimensionless axial coordinate], the fluid speed is set to zero at the tube wall ($\mathscr{R} = 1$), and boundedness is imposed at the tube center ($\mathscr{R} = 0$). Since at least one of C_1 and C_2 is arbitrary, the solution of the homogeneous system (5) is not unique, and the determinant of the coefficient matrix must be zero, yielding the dispersion relation

$$\begin{vmatrix} M(k)\, J_1(M(k)) & kJ_1(A(k)) \\ kJ_0(M(k)) & A(k)\, J_0(A(k)) \end{vmatrix} = 0. \tag{9}$$

Equation (9) is in the form of (2), with all coefficients involving complex transcendental functions of infinite order in k. Application of the Method of Eigenvalleys to (9) now yields all possible solutions for k as a function of parameters F and D.

The auxiliary function $E(k)$ is obtained by expanding the left-hand side of (9) for arbitrary complex k to yield

$$E(k) = MA\,\frac{J_1(M)}{J_0(M)} - k^2\,\frac{J_1(A)}{J_0(A)}, \tag{10}$$

where $E(k)$ has been normalized by the product $J_0(A)\, J_0(M)$ in order to give reasonable eigensurface heights away from the eigenvalleys.

Figures 3(a,b) and 4(a,b,c) show a typical set of the level curves of $|E(k)|$.[4] The

[4] The contour elevations increase logarithmically in height starting at contour number 2 with a height of 1.0, with successively higher elevations of 2, 3,..., 9, 10, 20,..., 90, 100, 200, etc.

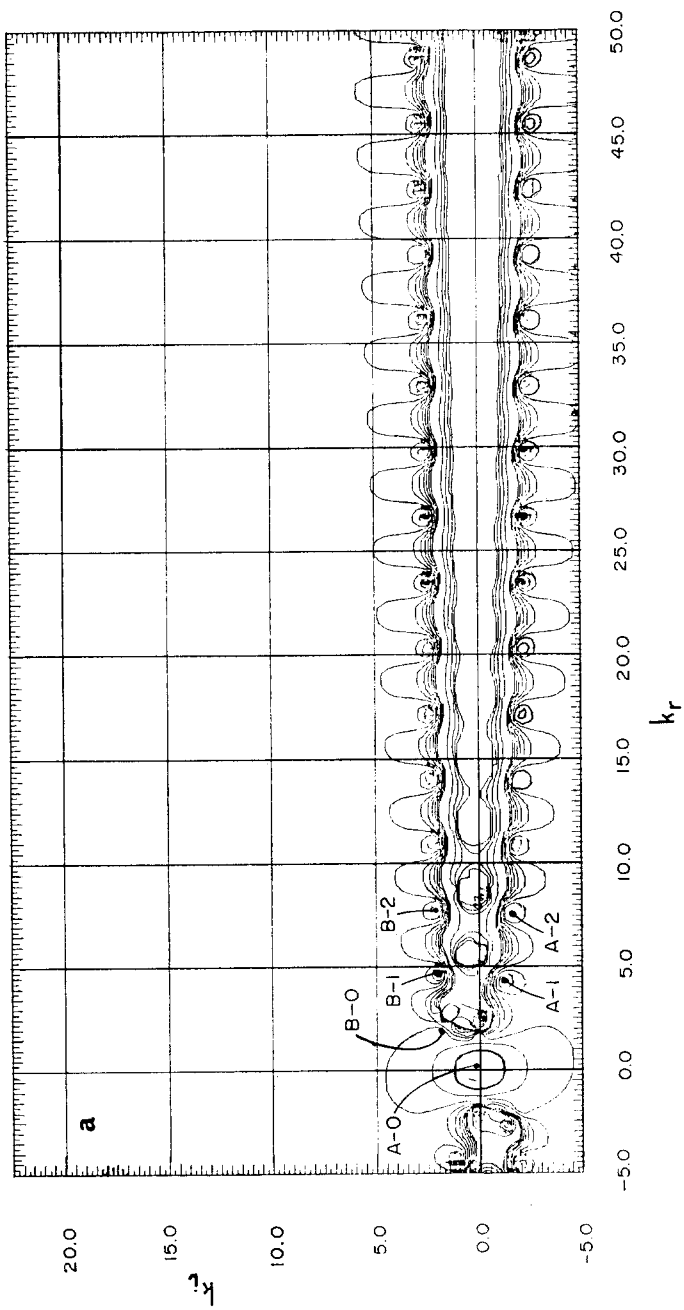

FIG. 3(a). Main eigensurface for $F = 0.08$ and $D = 0.01$.

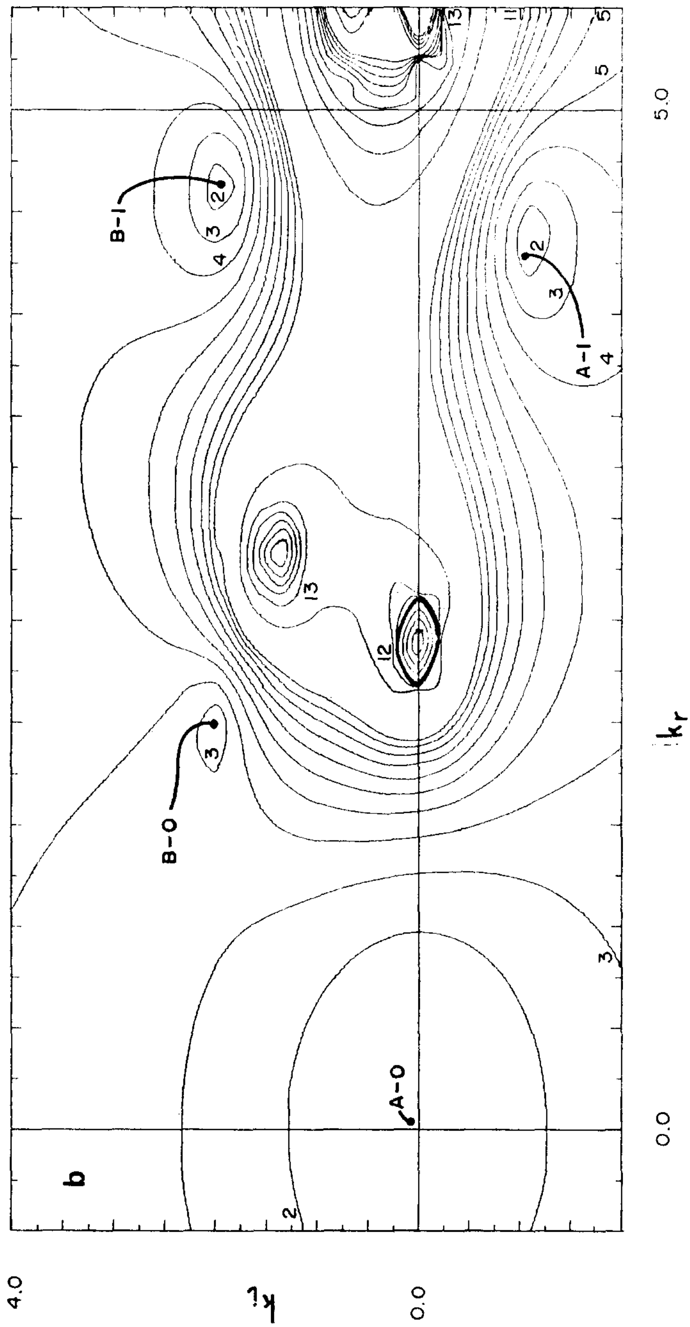

FIG. 3(b). Resolved and magnified eigensurface for $F = 0.08$ and $D = 0.01$.

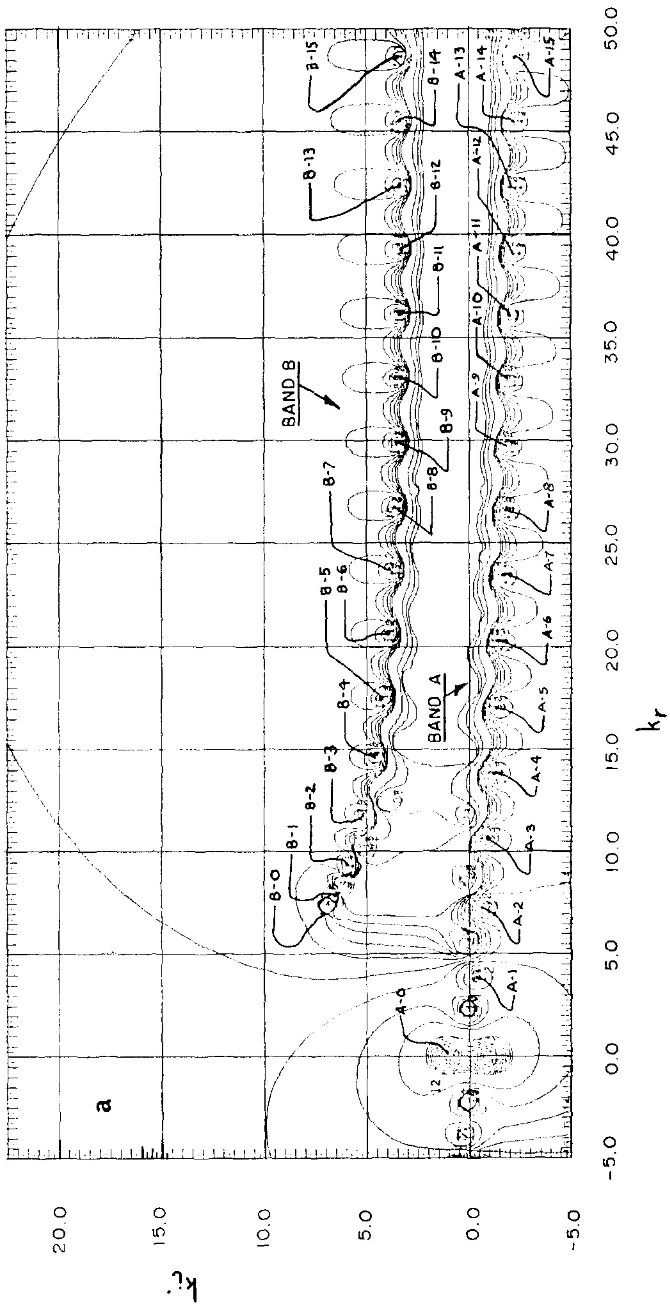

FIG. 4(a). Main eigensurface with eigenvalue numbering scheme defined for $F = 1.0$ and $D = 0.01$.

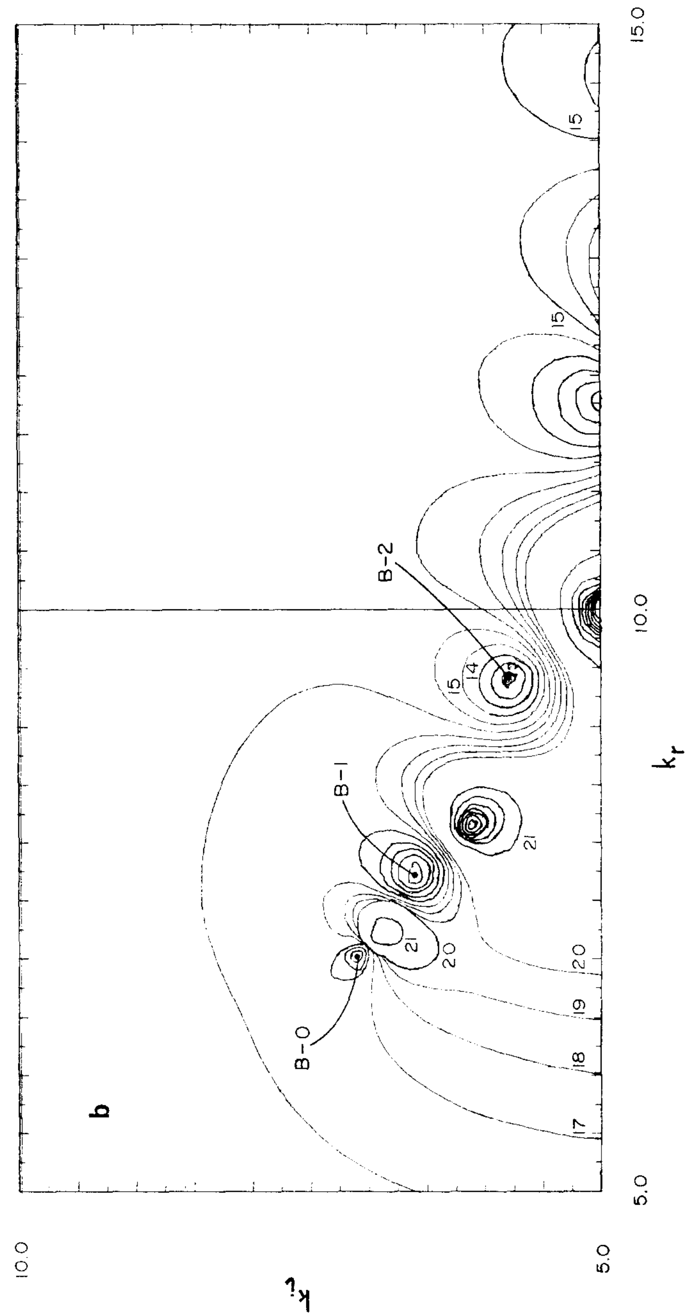

FIG. 4(b). Resolved and magnified eigensurface for $F = 1.0$ and $D = 0.01$.

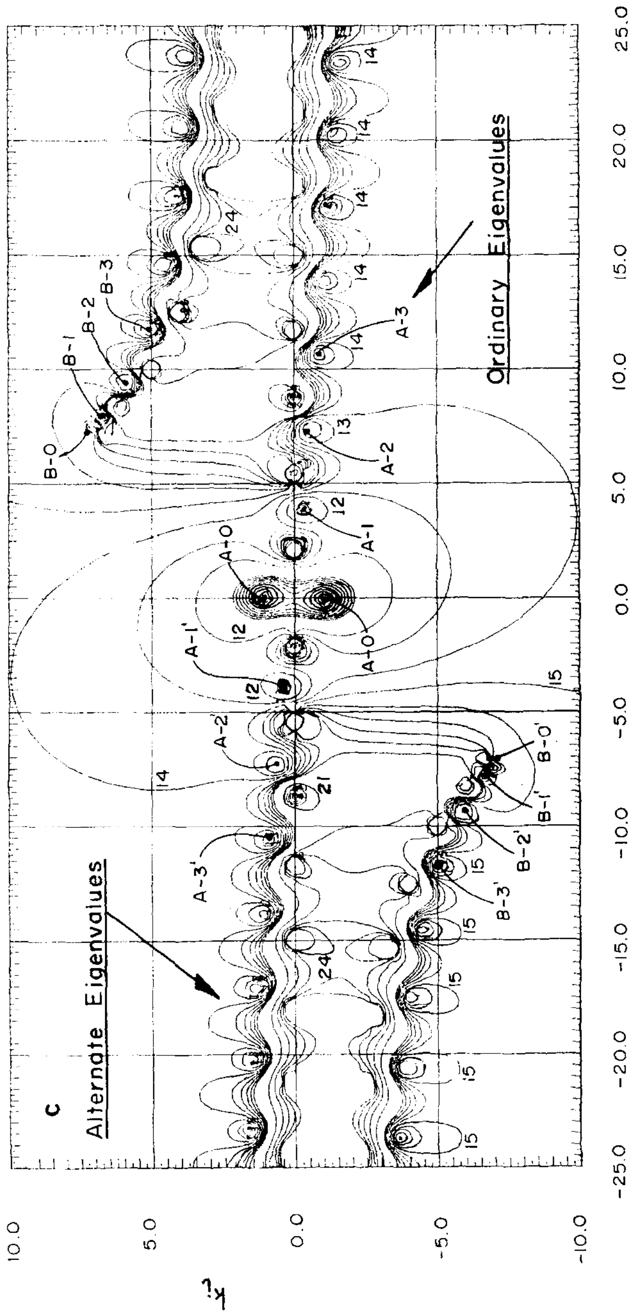

FIG. 4(c). Main eigensurface showing symmetry about the origin for $F = 1.0$ and $D = 0.01$.

coordinates of the k solution space for Figs. 3(a) and 4(a) are ($-5.0 \leqslant k_r \leqslant 50.0$) and ($-5.0 \leqslant k_i \leqslant 25.0$). The resolution of contour program grid points for these two plots is $\Delta k_r = \Delta k_i = 0.46$. An average eigensurface contour map involved the calculation of over 29 000 complex Bessel functions [3, 5], and took 15 min to produce on a Univac 1108 digital computer (add time 0.75 μsec), exclusive of off-line plot time on a Calcomp 663 incremental plotter [2, 3]. Figures 3(b) and 4(b) show cases in which magnification was necessary [because the average width of eigenvalley B–0 (0.2) was smaller than the grid size (0.46) used by the contour program, a situation analogous to that of eigenvalue k_4, Figs. 1 and 2].

A total of thirty-two eigenvalleys were identified in the k solution space; and their coordinates were fed to a pattern search minimization heuristic [3] for the accurate determination of the eigenvalues. The average run time required to compute all of these eigenvalues to five-place accuracy was 7 min. These eigenvalues are extensively tabulated in the dissertation by Scarton [3]. [Again, because of the narrow width of eigenvalley B–0, it was difficult to make the calculation converge completely, in this particular case. Eigenvalue k_{B-0} was later found to have the exact analytic form $k_{B-0} = \pm\sqrt{(F/2D)}(1+j)$.]

In Figs. 3 and 4, the eigenvalues are arranged in two bands, called the ordinary A and B bands, and two alternate bands of opposite sign, since $E(k)$ is an even function of k. Located between the eigenvalues are the poles resulting from the zeroes of artificially introduced normalizing functions $J_0(A)$ and $J_0(M)$.

Figure 4(c) indicates an important restriction on the Method of Eigenvalleys: that it cannot discriminate between the various sheets of the Riemann surface associated with (9). Thus, the variable separable solution (8) is derivable from

$$f(\mathscr{R}, Z, T) = \mathrm{Re}(f_1(\mathscr{R})\, e^{jFT+kZ}) + \mathrm{Re}(f_2(\mathscr{R})\, e^{jFT-kZ}), \tag{11}$$

where the $+k$ term refers to solutions of k on the first sheet of the two-sheeted Riemann surface, associated with the double-valued square root of k^2, and the $-k$ term refers to solutions of k on the second sheet. If the solutions on each of the two sheets are allowed to overlap onto one equivalent sheet, then (11) becomes the reduced form (8). Hence, Fig. 4(c) shows the joint solutions of both Riemann surfaces as the ordinary A and B bands and the alternate A and B bands, which for this problem are the negatives of each other.[5]

Figure 5 shows a summary of the eigenvalue trajectories for $D = 0.01$. Extension of the plotting domain to include larger values of k_r (e.g., $50.0 \leqslant k_r \leqslant 100.0$) demonstrates that infinitely many eigenvalues exist near the real axis; these well-

[5] For an example of when the symmetry of $|E(k)|$ about the origin does not occur, see the dissertation by Scarton [3, Chapter IV].

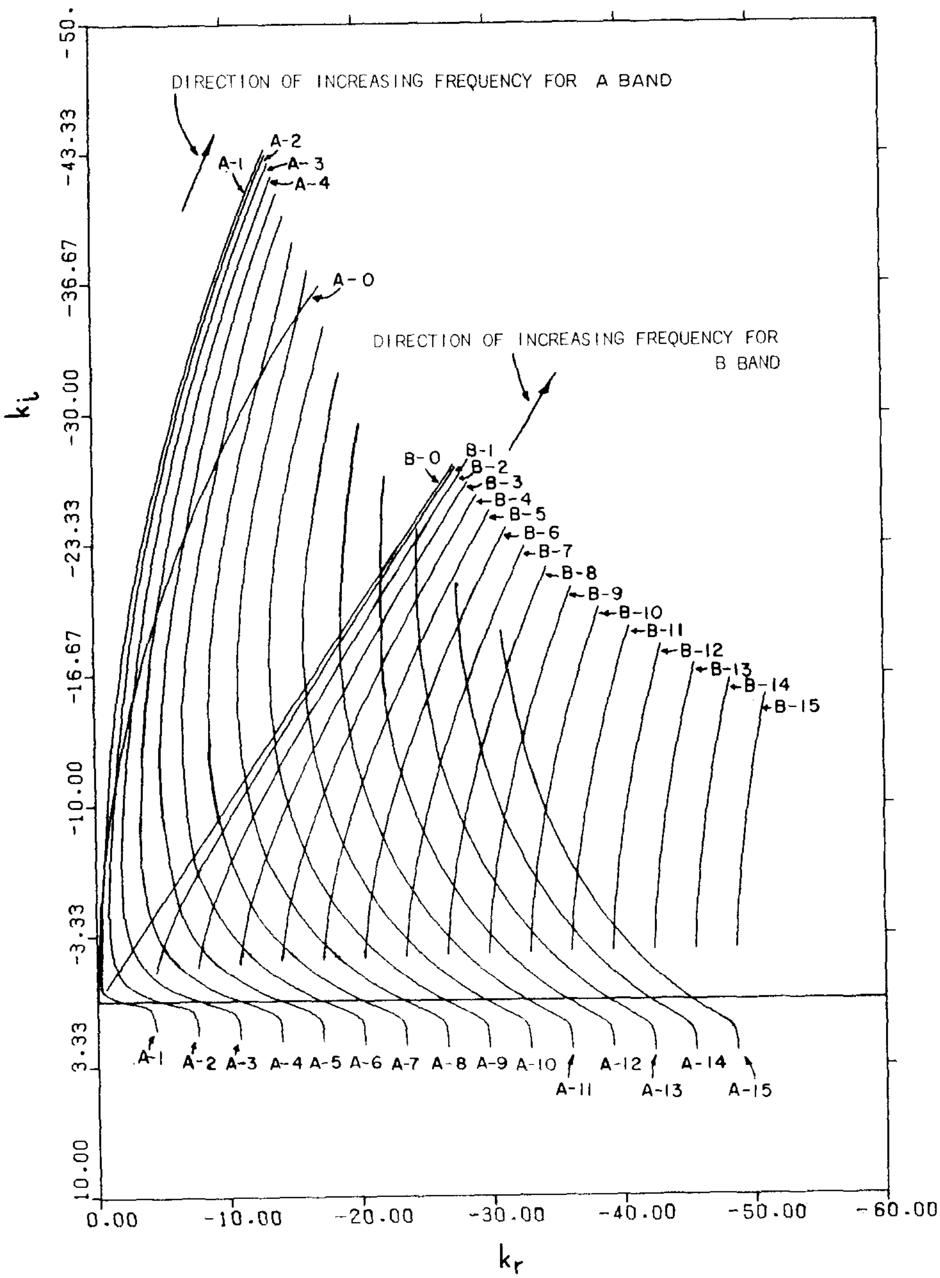

FIG. 5. Eigenvalue trajectories for $D = 0.01$ [A band frequency range $(0.01 \leqslant F \leqslant 50.0)$; B band frequency range $(0.01 \leqslant F \leqslant 15.0)$].

ordered complex conjugate equations have been found by Fitz-Gerald [6, 8] to satisfy the relation

$$k_n = \mp_{+}^{} (2n + 1) \frac{\pi}{2} \mp_{\pm} j \ln[(4n + 2)\pi]$$

for large n. Figure 5 also shows that eigenvalues k_{A-0} and k_{B-0} are all approaching an analytically verifiable degenerate state at the origin when $F = 0$.

As a final note, the Method of Eigenvalleys can also be used to solve the inverse temporal stability problem of finding the solutions of (9) for *real* wavenumber k and *complex* dimensionless frequency $F = F_r + jF_i$.

For a much expanded description of the Method of Eigenvalleys, see the dissertation by Scarton [3, Chapter II]. This expanded version gives many procedural details of this method which have been omitted here due to space limitations. Complete program descriptions, actual FORTRAN and ALGOL source program listings, and typical input data arrangement listings, are also included [3, pp. 588–594, 623–655, 664, 668–672, 677–685, 752–755]. These programs require only a FORTRAN real-function subroutine FNCTN, to return the value of $|E(k)|$ for given values of k_r, k_i, and system parameters.

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